A Useful Simple Recursion Formula to Account for Inelastic Collision Effects on the Electron Distribution in a Plasma

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The competitive effects of inelastic electron-atom collisions and electron-electron interactions on a sufficiently isotropic electron distribution is considered for a low temperature monatomic plasma. It is presupposed that unbalanced collisional transitions between ground state and resonance level of the plasma atoms are the dominant disturbance effect on the electron distribution. The unbalance is assumed to be due to resonance radiation escape and excitation diffusion. An analytical solution, which permits the calculation of the electron distribution even well above the ionization energy of the atomic ground state, is presented. In particular, it is demonstrated that due to the rapid decrease of the Coulomb cross section with increasing electron energy, the solution for the high energy tail of the electron distribution can be related to its form at lower energies by a simple recursion formula. The latter enables an easy computation of the ionization coefficients even for the lowest atomic energy levels. The analytical solution and its high energy approximation are numerically evaluated for the example of a low temperature–low pressure cesium plasma.

I. Introduction

In the theory of low pressure gas discharges the state of the electron gas, as reflected in the form of the electron distribution function, is of central importance. This is particularly so, because the deviation from the Maxwellian electron distribution is directly related to energy loss mechanisms such as radiation escape as well as excitation and ambipolar diffusion. Information regarding the state of the electron gas is, therefore, of practical relevance to technical discharge devices such as cesium diodes operated in the arc mode[1].

The effect of inelastic collisions on the isotropic part of the electron distribution in a collision dominated monatomic plasma has been studied by numerous authors[2–17]. Either numerical or approximate analytical solutions of the governing equations are presented. As far as they incorporate superelastic electron collisions, the analytical solutions are restricted to a rather limited electron energy range above the atomic resonance excitation energy.

In order to extend the analytical solution to higher energies and to approximate the high energy tail of the electron distribution by as simple an expression as practicable, we reconsider this problem for a monatomic model plasma with certain requirements regarding relaxation state, macroscopic parameter range, and internal structure of its components. The search for a high energy approximation to the electron distribution function led to the recovery of a simple recursion formula relating the distribution of high energy electrons to that of low energy electrons.

Although our method applies to many plasmas, we choose for our numerical evaluation a cesium plasma, for which, because of its great practical relevance, the most important cross section and oscillator strength data are now available.

In the following sections we give the model assumptions and present the mathematical equations together with the method of solution. The recursion model is formulated and applied to the calculation of the ground state ionization coefficient. The results are numerically evaluated and discussed.

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II. Model Assumptions

The following basic model assumptions specifying the plasma under consideration have been made:

1. The electron distribution function is nearly isotropic, its isotropic part being sufficiently homogeneous. This assumption restricts the treatment to collision dominated plasmas, in which

$$(1/l_-)\sqrt{\lambda^{\varepsilon}_{--}(\eta)/\sum_{\sigma=+,-}1/\lambda^{\mathrm{p}}_{-\sigma}(\eta)} \ll 1 \qquad (1a)$$

holds, where l_- is the characterisic length for spatial inhomogeneities in the macroscopic parameters of the electron gas and $\lambda^{\mathrm{p}}_{-\beta}(\eta)$ and $\lambda^{\varepsilon}_{-\beta}(\eta)$ denote respectively the momentum and the energy relaxation

length for electrons of energy $\varepsilon_- = kT_- \eta$ colliding with thermalized particles of the plasma component β.

2. The effect of electron-ion and elastic electron-atom collisions as well as the electric field effect on the isotropic part of the electron distribution is negligibly small compared to electron-electron interactions. This assumption imposes a lower limit on the ionization degree and an upper limit on the electric field strength. The limits are given by the inequalities *

$$\sum_{\sigma=+,0} 2\frac{m_-}{m_\sigma}\frac{\lambda^{\mathrm{p}}_{--}(\eta)}{\lambda^{\mathrm{p}}_{-\sigma}(\Delta)\eta} \ll 1 \tag{1b}$$

and

$$\frac{eE}{kT_-}\sqrt{\frac{\lambda^{\varepsilon}_{--}(\eta)}{\sum_{\sigma=+,0} 1/\lambda^{\mathrm{p}}_{-\sigma}(\eta)}} \ll 1 \tag{1c}$$

see Figure 1. Strictly speaking, squares and products of the left hand sides of (1a) and (1c) must be small compared to unity to ensure that the isotropic part of the electron distribution is only weakly disturbed by spatial inhomogeneities and electric fields.

3. The mean thermal electron energy is much less than the resonance excitation energy of the atomic species. This assumption restricts the theory to low temperature plasmas.

4. The effect of nonresonance transitions on the isotropic part of the electron distribution is negligibly small compared to both resonance transitions and electron-electron interactions. This assumption again demands a certain ionization degree and sufficiently low excitation temperatures.

5. The effect of transitions to and from higher levels and the continuum on the resonance level population density is negligibly small compared to the influence of transitions to and from the ground state. This assumption implies that transitions between resonance level and higher levels are either in detailed balance or much more improbable than transitions between resonance level and ground state. The latter are assumed to be unbalanced by radiation escape and excitation diffusion.

6. The electron density can be varied without a significant change in the radiation escape factor and the diffusion time of the resonance level. This assumption implies that nonresonance radiation escape and ambipolar diffusion can be varied nearly independent of the radiation escape and excitation diffusion of the resonance level.

* In the following the indices −, 0, 1 signify respectively electrons, ground state atoms, and atoms excited to the resonance level, where the fine structure is coalesced into one effective level. The only exception from this rule is ε_0, the dielectric constant of the vacuum. Primes indicate derivatives with respect to η. Many symbols have their usual meaning and need no further explanation.

III. Basic Equations

According to the above plasma model the problem reduces to a simultaneous solution of the spatially homogeneous kinetic equation of the electrons and the particle balance equation of the resonance level, where plasma pressure, gas temperature, density and kinetic temperature of the electrons, resonance radiation escape factor, and resonance excitation diffusion time are introduced as free parameters.

The kinetic equation of the electrons reduces to a sum of Fokker-Planck and Boltzmann collision terms. The standard procedure of an expansion of the electron distribution function in spherical harmonics produces a hierarchy of equations for the expansion coefficients. If the distribution function is nearly isotropic, the equation for the isotropic part, which is of interest in connection with the particle balance equations of the various atomic energy levels, can be isolated from the rest of the hierarchy. In terms of the normalized distribution function

$$\gamma(\eta) \equiv f_-(\eta)/f^{\mathrm{M}}_-(\eta)$$

$$\text{with} \quad f^{\mathrm{M}}_-(\eta) = n_-(m_-/2\pi kT_-)^{3/2}\exp(-\eta) \tag{2}$$

$$\text{and} \quad \eta \equiv \varepsilon_-/kT_-$$

the kinetic equation for the isotropic part of the electron distribution is given by

$$\begin{aligned}(\nu^*_{--}\exp(\eta)/\eta)\,[F(\eta)\,\gamma'(\eta)\exp(-\eta)]' \\ + n_0\,Q^{-1}_{-0}\,(\eta,\bar{\eta}_1)\,[\alpha_1\gamma(\eta-\bar{\eta}_1)-\gamma(\eta)] \\ + (1+\bar{\eta}_1/\eta)\,n_0\,Q^{-1}_{-0}\,(\eta+\bar{\eta}_1,\bar{\eta}_1)\,\exp(-\bar{\eta}_1) \\ \cdot[\gamma(\eta+\bar{\eta}_1)-\alpha_1\gamma(\eta)] = 0\end{aligned} \tag{3}$$

where Q^{-1}_{-0} and $\sqrt{(8kT_-/\pi m_-)}\,\nu^*_{--}$ are respectively the cross section for electron collisional excitation of the resonance level and the electron-electron interaction frequency. ν^*_{--} is defined by

$$\nu^*_{--} = 2\pi n_-(e^2/4\pi\varepsilon_0 kT_-)^2 \ln\Lambda_{--} \tag{4}$$

$$\text{with} \quad \Lambda_{--} = 12\pi/\sqrt{n_-(\varepsilon_0 kT_-/e^2)^3}$$

where the MKSA-system of units is used. The factor $F(\eta)$ arises from the integral coefficients of the

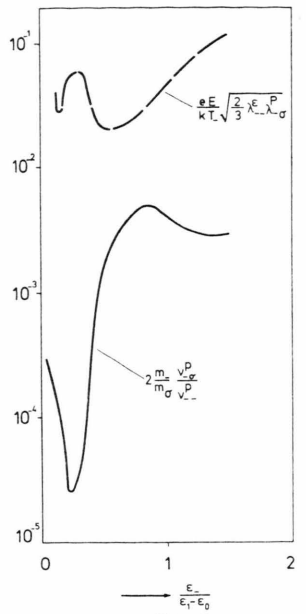

Fig. 1

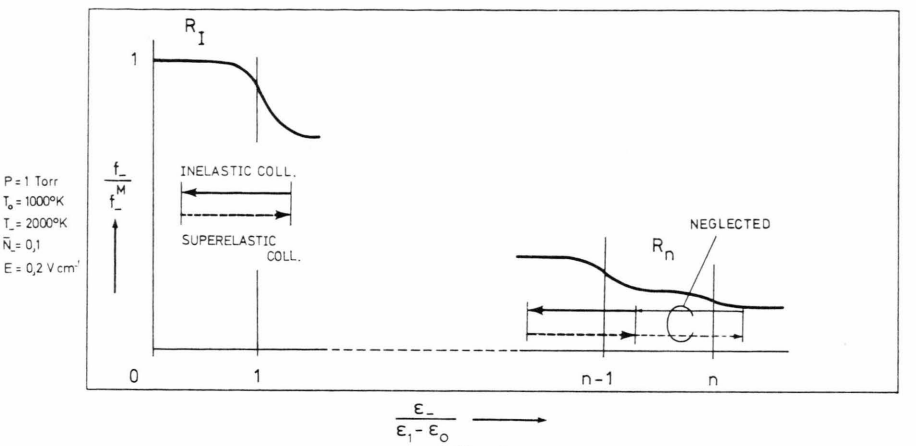

Fig. 2

Fig. 1. Characteristic quantities giving the relative influence (compared to that from electron-electron interactions) of elastic electron-atom collisions (solid line) and electric fields (dashed line) on the electron distribution in a cesium plasma. The electric field strength E is estimated by the formula

$$\frac{e\,E}{k\,T_-} \cong \sum_{\sigma \neq -} n_\sigma\, Q^{\mathrm{p}}_{-\sigma} \sqrt{\frac{128}{3\,\pi}\,\frac{m_-}{m_\sigma}\left(1-\frac{T_\sigma}{T_-}\right)}$$

for the parameter combinations used in the numerical calculations. The values of the electron-atom momentum transfer cross section Q^{p}_{-0} are taken from a paper by Nighan [18].

Fig. 2. R_n-subdivision of the energy range and characterization of the relevant processes.

Fokker-Planck collision term evaluated with a Maxwellian electron distribution.

The particle balance equation of the resonance level reduces to the sum of terms describing diffusion and collisional-radiative transitions to and from the ground state. In terms of the normalized level population density

$$\alpha_1 \equiv n_1/n_1^{\mathrm{B}} \text{ with } n_1^{\mathrm{B}} = n_0\,(g_1/g_0)\exp\,(-\bar{\eta}_1) \quad \text{and}$$
$$\bar{\eta}_1 \equiv (\epsilon_1-\epsilon_0)/kT_- \tag{5}$$

one obtains the equation

$$\alpha_1\,(1/\tau_{1\mathrm{d}} + A_{1\to 0}\,A_{1\to 0}) \cong \frac{n_-\sqrt{8\,kT_-/\pi\,m_-}}{(g_1/g_0)\exp\,(-\bar{\eta}_1)} \tag{6}$$
$$\cdot \int_{\bar{\eta}_1}^{\infty} [\gamma(\eta) - \alpha_1\,\gamma\,(\eta-\bar{\eta}_1)\,]\;\eta\;Q^{-1}_{-0}(\eta,\bar{\eta}_1)\exp(-\eta)\,\mathrm{d}\eta$$

where the unbalance between electron collisional transitions due to radiation escape and excitation diffusion is described by the e cape factor $A_{1\to 0}$ [19] and the diffusion time $\tau_{1\mathrm{d}}\cdot A_{1\to 0}$ and g_1/g_0 are respective'y the Einstein coefficient for spontaneous emission of resonance radiation and the ratio between the statistical weights o resonance level and ground state. The Einstein coefficient is related to the absorption oscillator strength $f_{0\to 1}$ by the equation

$$A_{1\to 0} = \frac{(e\,k\,T_-\,\bar{\eta}_1)^2}{2\,\pi\,\varepsilon_0\,\hbar^2\,c^3\,m_-}\cdot\frac{g_0}{g_1}\,f_{0\to 1}\,. \tag{7}$$

For cesium, $f_{0\to 1} \cong 1.13$ as given by Anderson and Zilitis [20], which is in sufficient agreement with values calculated by Stone [21] for the esonance doublet.

To enab'e an analytical solution of the governing equa ions, the cross section for electron collisional excitation of the resonance level has been approximated by the formula

$$Q^{-1}_{-0}\,(\eta,\bar{\eta}_1) = S_{0\to 1}kT_-\,\bar{\eta}_1(1-\bar{\eta}_1/\eta) \tag{8}$$

which, when using the value $S_{0\to 1} \cong 95$ Å^2/eV, correctly describes the threshold slope of the experimental cross section for cesium resonance excitation as found by Zapesochnyi and Shimon [22].

IV. Energy Regime Subdivision

In order to solve Eqs. (3) and (6) simultaneously, it is necessary to subdivide the electron energy regime into multiples of the resonance excitation energy (see Figure 2). In the first subregion R_I below the resonance excitation energy only inelastic electron energy transitions to and from higher energies are possible. In all other subregions (typically R_n) both transitions to and from higher energies as well as to and from lower energies occur. However, for kinetic electron temperatures low in comparison to the resonance excitation energy, transitions to and from higher energies can be neglected compared to transitions to and from lower energies. This fact results from the exponentially decreasing number of electrons with increasing electron energy. Mathematically, this amounts to the inequality

$$\left(1+\frac{\bar{\eta}_1}{\eta}\right)\frac{Q_{-0}^{-1}(\eta+\bar{\eta}_1,\bar{\eta}_1)}{Q_{-0}^{-1}(\eta,\bar{\eta}_1)\exp\bar{\eta}_1}\ll 1\,. \tag{9}$$

From the electron energy regime subdivision and the inequality (9) the following reduced system of coupled equations for $\gamma(\eta)$ and α_1 results:

$$\begin{aligned}&[F(\eta)\,\gamma_\mathrm{I}'(\eta)\exp(-\eta)]' = \varphi(\eta) \quad \text{for} \quad 0\leqq\eta\leqq\bar{\eta}_1\\ &\qquad \gamma_n''(\eta)-\gamma_n'(\eta)\\ &\quad +\varkappa_1(\eta-\bar{\eta}_1)\,[\alpha_1\gamma_{n-1}(\eta-\bar{\eta}_1)-\ \gamma_n(\eta)]=0\,,\end{aligned} \tag{10}$$

$$\begin{aligned}\alpha_1 = -\,\bar{\alpha}_1\exp\,(-\bar{\eta}_1)\,\gamma_\mathrm{I}'(\bar{\eta}_1)\ \mid\ &\text{for}\ (n-1)\,\bar{\eta}_1\leqq\eta\\ &\leqq n\,\bar{\eta}_1\,;\quad n\geqq 2\end{aligned} \tag{11}$$

where $\varphi(\eta)$ is to be defined in connection with the iterative solution scheme. $\bar{\alpha}_1$ and $\varkappa_1$ are given by the expressions

$$\bar{\alpha}_1 \equiv \frac{\nu_{--}^*\,n_-\sqrt{8\,kT_-/\pi\,m_-}}{n_0(g_1/g_0)\exp\,(-\bar{\eta}_1)\,(1/\tau_{1\mathrm{d}}+A_{1\to 0}A_{1\to 0})}$$

and

$$\varkappa_1 \equiv n_0\,S_{0\to 1}\,kT_-\,\bar{\eta}_1/\nu_{--}^*\,. \tag{12}$$

To guarantee the existence of the second derivative of the function $\gamma(\eta)$ for all values of its argument η, the boundary conditions

$$\begin{aligned}\gamma_{n-1}((n-1)\,\bar{\eta}_1) &= \gamma_n((n-1)\,\bar{\eta}_1)\,,\\ \gamma'_{n-1}((n-1)\,\bar{\eta}_1) &= \gamma_n'((n-1)\,\bar{\eta}_1)\end{aligned} \tag{13}$$

are required. Physically, these conditions guarantee a unique electron density and a steady electron flux in energy space for all values of η. The boundary condition at infinity is

$$\lim_{\eta\to\infty}\gamma(\eta)<1 \tag{14}$$

consistent with our plasma model, in which only electron energy loss mechanisms are considered. The boundary conditions are supplemented by the normalization condition

$$2/\sqrt{\pi}\int_0^{\eta_1}\sqrt{\eta}\,\gamma_\mathrm{I}(\eta)\exp(-\eta)\,\mathrm{d}\eta\cong 1\,. \tag{15}$$

For given electron temperature and resonance excitation energy the solution to Eqs. (10) and (11) depends on two basic characteristic quantities: $\bar{\alpha}_1$, which is the ratio between electron-electron interaction rate to resonance level deexcitation rate due to radiation escape and excitation diffusion, and $\varkappa_1$, which is the ratio between inelastic electron-atom collision rate to electron-electron interaction rate. For the assumed plasma model, $\bar{\alpha}_1\to\infty$ yields the equilibrium solution $\gamma(\eta)=1$ and $\alpha_1=1$, independent of the value $\varkappa_1$.

Equations (10) and (11) are valid within the validity range of Equation (8). It is assumed that Eq. (8) is still applicable in the environment of the ground state ionization energy of the atomic species. In view of the limited validity range of the given cross section formula and because of the increasing complexity of the analytical solution with increasing electron energy, the solution procedure is terminated in the subregion just above the ground state ionization energy. The last subregion is formally extended to infinity to incorporate the boundary condition, Equation (14). Consequently, the solution for the last subregion must depend on an insufficiently defined function. This fact causes a precision loss in the evaluation of the analytical expressions in the high energy range, as is shown below.

In view of its application to a cesium plasma, the solution is formulated for the first four subregions only.

V. Iterative Solution Scheme

An iterative procedure has been adopted for the solution of the governing equations*. The lowest approximation $\gamma^{(0)}(\eta)$ and $\alpha_1^{(0)}$ of the iterative scheme results from a complete neglect of all inelastic and superelastic collisions in the first subregion R_I of the electron energy scale. This means that in the lowest approximation the electron

* This procedure was also applied by Shaw, Mitchner, and Kruger [9].

distribution function is assumed in such a way that $\varphi(\eta) = 0$.

In the first order approximation, $\gamma^{(1)}(\eta)$ and $\alpha_1{}^{(1)}$, the inelastic and superelastic electron collisions in R_I are evaluated in the zero order approximation of the electron distribution function and the resonance level population, i.e. for

$$\varphi(\eta) = \varphi^{(0)}(\eta) \equiv -\varkappa_1 \eta \cdot \cdot \exp(-\eta - \bar{\eta}_1)\,[\gamma_\mathrm{II}{}^{(0)}\ (\eta + \bar{\eta}_1) - \alpha_1{}^{(0)}\gamma_\mathrm{I}{}^{(0)}(\eta)]\,. \tag{16}$$

Proceeding, to obtain the n-th order approximation for the electron distribution function, one uses the $(n-1)$ st form of the function $\varphi(\eta)$, which in turn contains the $(n-1)$ st approximation to the electron distribution function. Also in all approximations except the lowest one, where $F(\eta) = 1$, the energy dependence of the integral coefficients of the Fokker-Planck collision term evaluated for a Maxwellian electron distribution is considered such that

$$F(\eta) = \mathrm{erf}\sqrt{\eta} - 2\sqrt{\eta/\pi}\exp(-\eta)\,. \tag{17}$$

VI. Zero Order Approximation

The analytical solution for $\gamma^{(0)}(\eta)$ and $\alpha_1{}^{(0)}$ leads to the following expressions in the subregions R_I to R_IV (see, for instance, Kamke[23] for the solution method)

$$\left.\begin{aligned}
\gamma_\mathrm{I}{}^{(0)}(\eta) &= 1 + K_1{}^{(0)}(\exp\eta - 1)\,,\\
\gamma_\mathrm{II}{}^{(0)}(\eta) &= \exp(\eta/2)\,[K_2{}^{(0)}\,\mathrm{Ai}(\xi) + K_3{}^{(0)}\,\mathrm{Bi}(\xi)]\\
&\quad + \alpha_1{}^{(0)}\gamma_\mathrm{I}{}^{(0)}\,(\eta - \bar{\eta}_1)\ ,\\
\gamma_\mathrm{III}{}^{(0)}(\eta) &= \exp(\eta/2)\,\mathrm{Ai}(\xi)\left[\frac{\gamma_\mathrm{II}{}^{(0)}\,(2\,\bar{\eta}_1)}{\exp\bar{\eta}_1\ \mathrm{Ai}(\xi_2)}\right.\\
&\quad \left. + \int_{\xi_2}^{\xi}\frac{\mathrm{d}\sigma}{\mathrm{Ai}^2(\sigma)}\,(\zeta^{(0)} + \int_{\xi_2}^{\sigma}\Phi^{(0)}(\tau)\,\mathrm{Ai}(\tau)\,\mathrm{d}\tau)\right],\\
\gamma_\mathrm{IV}{}^{(0)}(\eta) &= \exp(\eta/2)\,\mathrm{Ai}(\xi)\left[\frac{\gamma_\mathrm{III}{}^{(0)}\,(3\,\bar{\eta}_1)}{\exp(3\,\bar{\eta}_1/2)\,\mathrm{Ai}(\xi_3)}\right.\\
&\quad \left. - \int_{\xi_3}^{\xi}\frac{\mathrm{d}\sigma}{\mathrm{Ai}^2(\sigma)}\int_{\sigma}^{\infty}\Psi^{(0)}(\tau)\,\mathrm{Ai}(\tau)\,\mathrm{d}\tau\right],
\end{aligned}\right\} \tag{18}$$

$$\alpha_1{}^{(0)} = -\bar{\alpha}_1 K_1{}^{(0)} \tag{19}$$

where $\zeta^{(0)} \equiv$

$$\frac{\gamma_\mathrm{II}{}^{(0)\prime}(2\,\bar{\eta}_1) - [\tfrac{1}{2} + \varkappa_1{}^{1/3}\mathrm{Ai}'(\xi_2)/\mathrm{Ai}(\xi_2)]\,\gamma_\mathrm{II}{}^{(0)}(2\,\bar{\eta}_1)}{\varkappa_1{}^{1/3}\exp\bar{\eta}_1\,/\mathrm{Ai}(\xi_2)}$$

$\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$ are Airy functions defined as solution of the differential equation $f''(\xi) - \xi f(\xi) = 0$ (see Abramowitz and Stegun[24]), where in the present case

$$\xi \equiv \xi_1 + \varkappa_1{}^{1/3}\,(\eta - \bar{\eta}_1) \quad \text{with} \quad \xi_1 \equiv 1/(4\varkappa_1{}^{2/3})$$

and $$\xi_n = \xi_1 + \varkappa_1{}^{1/3}(n-1)\,\bar{\eta}_1\,. \tag{20}$$

The normalization condition and parts of the boundary conditions have already been incorporated in the special form of this solution. In particular, the condition of a finite solution for $\eta \to \infty$ has been incorporated by the infinite limit of integration in the solution for the subregion R_IV, which causes a precision loss in the evaluation, because the auxiliary function $\psi^{(0)}(\tau)$ is not known for high values of its argument:

$$\psi^{(0)}(\tau)\begin{cases}
\equiv -\alpha_1{}^{(0)}\exp(-\bar{\eta}_1/2)\,(\tau - \xi_1)\\
\cdot\exp[-(\tau-\xi_1)/2\varkappa_1{}^{1/3}]\,\gamma_\mathrm{III}{}^{(0)}[(\tau-\xi_1)/\varkappa_1{}^{1/3}]\\
\text{for} \qquad \xi_3 \leqq \xi \leqq \xi_4\,,\\
\text{unknown for} \quad \xi > \xi_4\,.
\end{cases} \tag{21}$$

Therefore the solution in R_IV is sufficiently accurate only up to those values of ξ, for which the inequality

$$\begin{aligned}
\left|\int_{\xi_4}^{\infty}\psi^{(0)}(\tau)\,\mathrm{Ai}(\tau)\mathrm{d}\tau\right| &\leq \alpha_1{}^{(0)}\gamma_\mathrm{III}{}^{(0)}\ (3\,\bar{\eta}_1)\exp(-\bar{\eta}_1/2)\\
&\quad\times\int_{\xi_4}^{\infty}(\tau-\xi_1)\exp\left[-\frac{(\tau-\xi_1)}{2\varkappa_1{}^{1/3}}\right]\mathrm{Ai}(\tau)\,\mathrm{d}\tau\\
&\ll\left|\int_{\xi}^{\xi_4}\Phi^{(0)}(\tau)\,\mathrm{Ai}(\tau)\,\mathrm{d}\tau\right|
\end{aligned} \tag{22}$$

is guaranteed, where the auxiliary function $\Phi^{(0)}(\tau)$ is defined by

$$\Phi^{(0)}(\tau) \equiv -\alpha_1{}^{(0)}\exp\ (-\bar{\eta}_1/2)\,(\tau - \xi_1) \cdot\exp\left(-\frac{\tau-\xi_1}{2\varkappa_1{}^{1/3}}\right)\gamma_\mathrm{II}{}^{(0)}\left(\frac{\tau-\xi_1}{\varkappa_1{}^{1/3}}\right)$$

for $$\xi_2 \leqq \xi \leqq \xi_3\,. \tag{23}$$

The remaining boundary conditions for $\gamma^{(0)}(\eta)$ allow one to calculate the integration constants $K_1{}^{(0)} \ldots K_3{}^{(0)}$ by a system of nonlinear algebraic equations. The equations resulting from the boundary conditions at $\eta = \bar{\eta}_1$ can be linearized, if the inequalities

$$\bar{\alpha}_1\exp(-\bar{\eta}_1/2)\,a_{12}a_{22}/\Delta^2 \ll 1 \quad \text{with} \quad \Delta \equiv \mathrm{Det}\,|a_{ij}| \tag{24}$$

and $$|a_{i3}K_3{}^{(0)}| \ll |a_{ij}K_j{}^{(0)}|$$

for $i = 1, 2$ and $j = 1, 2$

hold, where

$$a_{1k} = \begin{bmatrix} \exp \bar{\eta}_1 - 1 + \bar{\alpha}_1 \\ -\operatorname{Ai}(\xi_1) \exp(\bar{\eta}_1/2) \\ -\operatorname{Bi}(\xi_1) \exp(\bar{\eta}_1/2) \end{bmatrix} \quad \text{and} \quad a_{2k} = \begin{bmatrix} \exp(\bar{\eta}_1/2) \\ -\tfrac{1}{2}\operatorname{Ai}(\xi_1) - \varkappa_1^{1/3}\operatorname{Ai}'(\xi_1) \\ -\tfrac{1}{2}\operatorname{Bi}(\xi_1) - \varkappa_1^{1/3}\operatorname{Bi}'(\xi_1) \end{bmatrix}. \tag{25}$$

From the linearized equations the simple relations

$$K_1^{(0)} = -a_{22}/\Delta \quad \text{and} \quad K_2^{(0)} = a_{21}/\Delta \tag{26}$$

follow. Once the integration constants $K_1^{(0)}$ and $K_2^{(0)}$ have been determined by Eq. (26), $\alpha_1^{(0)}$ and $K_3^{(0)}$ can be calculated by Eq. (19) and the boundary conditions at $\eta = 3\,\bar{\eta}_1$, from which one obtains

$$K_3^{(0)} = \frac{b_3 - \sum_{k=1}^{2} a_{3k} K_k^{(0)}}{a_{33}} \tag{27}$$

where

$$a_{3k} \cong \begin{bmatrix} \alpha_1^{(0)} \left[\tfrac{1}{2} - \varkappa_1^{1/3} \dfrac{\operatorname{Ai}'(\xi_2)}{\operatorname{Ai}(\xi_2)} \right] \\ -\dfrac{\alpha_1^{(0)} \varkappa_1^{1/3} \exp(-\bar{\eta}_1/2)}{\operatorname{Ai}(\xi_2)} \int_{\xi_2}^{\xi_3} \operatorname{Ai}(\tau)\,(\tau - \xi_1)\operatorname{Ai}(\tau - \varkappa_1^{1/3}\bar{\eta}_1)\,\mathrm{d}\tau \\ \dfrac{\varkappa_1^{1/3}}{\pi \operatorname{Ai}(\xi_2)} - \dfrac{\alpha_1^{(0)} \varkappa_1^{1/3} \exp(-\bar{\eta}_1/2)}{\operatorname{Ai}(\xi_2)} \int_{\xi_2}^{\xi_3} \operatorname{Ai}(\tau)\,(\tau - \xi_1)\operatorname{Bi}(\tau - \varkappa_1^{1/3}\bar{\eta}_1)\,\mathrm{d}\tau\,, \end{bmatrix} \tag{28}$$

$$b_3 \cong \frac{\alpha_1^{(0)}(1 - \alpha_1^{(0)})}{\exp \bar{\eta}_1} \left[\tfrac{1}{2} + \varkappa_1^{1/3} \frac{\operatorname{Ai}'(\xi_2)}{\operatorname{Ai}(\xi_2)} \right]. \tag{29}$$

In the derivation of a_{31} and b_3 use has been made of the inequalties

$$\alpha_1^{(0)} \exp(-\bar{\eta}_1) \ll 1,$$
$$\alpha_1^{(0)} \exp(-\bar{\eta}_1/2)\operatorname{Ai}(\xi_3)/\operatorname{Ai}(\xi_2) \ll 1, \text{ and}$$
$$\alpha_1^{(0)} \exp(-\bar{\eta}_1/2)\operatorname{Ai}'(\xi_3)/\operatorname{Ai}'(\xi_2) \ll 1.$$

For the parameter values used for the numerical evaluation of the solution formulae the inequalities (24) and consequently the simple relations, Eqs. (26), turned out to be valid.

VII. First Order Approximation

Introducing the value $\varphi(\eta) = \varphi^{(0)}(\eta)$ into Eq. (10) one obtains the first order approximation of the iterative solution scheme. For R_{I} the solution reads

$$\gamma_{\mathrm{I}}^{(1)}(\eta) = K_0^{(1)} + \int_0^{\eta} \frac{\exp\sigma}{F(\sigma)} \left[K_1^{(1)} + \int_0^{\sigma} \varphi^{(0)}(\tau)\,\mathrm{d}\tau \right] \mathrm{d}\sigma\,, \tag{30}$$

$$\alpha_1^{(1)} = -\bar{\alpha}_1 \exp(-\bar{\eta}_1)\,\gamma_{\mathrm{I}}^{(1)\prime}(\bar{\eta}_1)\,.$$

In order to satisfy the normalization condition, Eq. (15), and to ensure a finite value of the derivative $\gamma_{\mathrm{I}}^{(1)\prime}(0)$, one must require $K_0^{(1)} \cong 1$ and $K_1^{(1)} = 0$. A straightforward integration leads to the equality

$$\int_0^{\sigma} \varphi^{(0)}(\tau)\,\mathrm{d}\tau = -K_2^{(0)} \exp(-\bar{\eta}_1/2)\,a_{22} \left[1 + \frac{\operatorname{Ai}(\xi_1 + \varkappa_1^{1/3}\sigma) + 2\varkappa_1^{1/3}\operatorname{Ai}'(\xi_1 + \varkappa_1^{1/3}\sigma)}{2\,a_{22}\exp(\sigma/2)} \right]$$
$$- K_3^{(0)} \exp(-\bar{\eta}_1/2)\,a_{23} \left[1 + \frac{\operatorname{Bi}(\xi_1 + \varkappa_1^{1/3}\sigma) + 2\varkappa_1^{1/3}\operatorname{Bi}'(\xi_1 + \varkappa_1^{1/3}\sigma)}{2\,a_{23}\exp(\sigma/2)} \right]. \tag{31}$$

By going to the limit $\sigma \to 0$ in the integrant functions, Eq. (31), it can be shown that $\gamma_{\mathrm{I}}^{(1)\prime}(0) = 0$ compared to $\gamma_{\mathrm{I}}^{(0)\prime}(0) = K_1^{(0)} \ll 1$.

The solution for R_{II} is

$$\gamma_{\mathrm{II}}^{(1)}(\eta) = \exp(\eta/2)\operatorname{Ai}(\xi) \left[K_2^{(1)} + \int_{\xi_1}^{\xi} \frac{\mathrm{d}\sigma}{\operatorname{Ai}^2(\sigma)} \left(K_3^{(1)} + \int_{\xi_1}^{\sigma} \Theta^{(1)}(\tau)\operatorname{Ai}(\tau)\mathrm{d}\tau \right) \right] \tag{32}$$

with the auxiliary function

$$\theta^{(1)}(\tau) \equiv -\alpha_1^{(1)} \exp(-\bar{\eta}_1/2)(\tau - \xi_1)\exp\left(-\frac{\tau-\xi_1}{2\varkappa_1^{1/3}}\right)\gamma_{\mathrm{I}}^{(1)}\left(\frac{\tau-\xi_1}{\varkappa_1^{1/3}}\right) \text{ for } \xi_1 \leqq \xi \leqq \xi_2 . \quad (33)$$

The integration constants $K_2^{(1)}$ and $K_3^{(1)}$ result from the boundary conditions at $\eta = \bar{\eta}_1$.

The first order approximations of the solution in R_{III} and R_{IV} have the same structure as the corresponding zero order approximations and are not repeated here.

VIII. Recursion Formula

Inspection of the differential equations (10) shows that for $\varkappa_1(\eta - \bar{\eta}_1) \gg 1$ the differential terms lose their decisive role in the determination of the solution. More generally, if

$$\eta\, n_0\, Q_{-0}^{\overset{-1}{\uparrow}}(\eta, \bar{\eta}_1)/\nu_{--}^{*} \gg 1 \quad (34)$$

the inelastic electron-atom collision frequency dominates the electron-electron interaction frequency, and electron diffusion in energy space loses its influence on the electron distribution function. Since the Coulomb cross section decreases with increasing electron energy, such a situation is expected to occur in the high electron energy range. Therefore, it should be possible to balance inelastic and superelastic collisions with one another in the high electron energy tail. This means that the complicated analytical expressions can be replaced by the simple recursion formula

$$\bar{\gamma}(\eta) = \alpha_1 \gamma(\eta - \bar{\eta}_1) . \quad (35)$$

In other words, the solution can be continued into the high electron energy range, once it has been formulated for the low electron energy range. Note that this is only possible within the validity range of the inequality (34) irrespective of the special form of the collisional excitation cross section $Q_{-0}^{\overset{-1}{\uparrow}}(\eta, \bar{\eta}_1)$.

IX. Applications

The recursion formula can be usefully applied to the evaluation of the ground state ionization coefficient

$$K_{0\to+} = \sqrt{\frac{8\,k\,T_-}{\pi\, m_-}} \int_{\eta_+}^{\infty} \gamma(\eta) \exp(-\eta)\, \eta\, Q_{-0}^{\overset{--+}{\uparrow}}(\eta, \bar{\eta}_+)\, \mathrm{d}\eta \quad (36)$$

which depends on the high energy tail of the electron distribution function. No malizing the ground state ionization coefficient to its equilibrium value and introducing the formula

$$Q_{-0}^{\overset{--+}{\uparrow}}(\eta, \bar{\eta}_+) = S_{0\to+}\, k\, T_-\, \bar{\eta}_+ (1 - \bar{\eta}_+/\eta) , \quad (37)$$

for the threshold region of the ground state ionization cross section one obtains the expression

$$K_{0\to+}^{(0)}/K_{0\to+}^{\mathrm{M}} = \alpha_1^{(0)} \exp(\bar{\eta}_+ - \bar{\eta}_1) \int_{\eta_+ - \eta_1}^{\infty} \gamma^{(0)}(\eta) \times \exp(-\eta)\,(\eta - \bar{\eta}_+ + \bar{\eta}_1)\, \mathrm{d}\eta \quad (38)$$

where the recursion formula has been applied to the zero order analytical solution. By an integral transformation the lower limit of integration is shifted to subregion R_{II}.

Moreover, by application of the Gryzinski formula[25]

$$Q_{-1}^{\overset{--+}{\uparrow}}(\eta, \bar{\eta}_+ - \bar{\eta}_1) = \frac{C\, q[\eta/(\bar{\eta}_+ - \bar{\eta}_1)]}{(k\,T_-)^2\,(\bar{\eta}_+ - \bar{\eta}_1)^2},$$
$$q(x) \equiv \frac{1}{x}\left(\frac{x-1}{x+1}\right)^{3/2} \cdot \left[1 + \frac{2}{3}\left(1 - \frac{1}{2x}\right)\ln(2.7 + \sqrt{x-1})\right] \quad (39)$$

for the resonance level ionization cross section, one can estimate the relative importance of resonance level and ground state ionization rate by the expression

$$\frac{I_{1\to+}^{(0)}}{I_{0\to+}^{(0)}} = \frac{C}{(g_0/g_1) S_{0\to+} (k\,T_-)^3\, \bar{\eta}_+\, (\bar{\eta}_+ - \bar{\eta}_1)^2} \cdot \frac{\int_{\eta_+ - \eta_1}^{\infty} \gamma^{(0)}(\eta) \exp(-\eta)\, \eta\, q[\eta/(\bar{\eta}_+ - \bar{\eta}_1)]\, \mathrm{d}\eta}{\int_{\eta_+ - \eta_1}^{\infty} \gamma^{(0)}(\eta) \exp(-\eta)\,(\eta - \bar{\eta}_+ + \bar{\eta}_1)\, \mathrm{d}\eta} . \quad (40)$$

Again, the zero order approximation of the iterative scheme and an integral transformation to subregion R_{II} have been used. For cesium, $C = 6.56\ 10^2 \text{Å}^2\text{eV}^2$ and $S_{0\to+} \cong 2.7\ \text{Å}^2/\text{eV}$ according to measurements by Nygaard[26] as well as by Zapesochnyi and Aleksakhin[27].

X. Numerical Procedure

For the numerical evaluation of the solution functions $\gamma^{(i)}(\eta)$ for $i = 0,1$, several transformations and approximation methods have been applied to save computer time and to avoid overflows in the machine calculation. In particular, partial integration by use of the Wronskian determinant of the Airy functions

$$\begin{vmatrix} \mathrm{Ai}(\xi) & \mathrm{Bi}(\xi) \\ \mathrm{Ai}'(\xi) & \mathrm{Bi}'(\xi) \end{vmatrix} = \frac{1}{\pi} \tag{41}$$

was performed to reduce the multiple integration. The latter turned out to be a particular time consuming factor in the machine calculation, since the Airy functions themselves were represented by integrals. In the high energy part of the analytical solution it was necessary to use the asymptotic expansions for products of Airy functions[24]

$$\mathrm{Ai}(x)\,\mathrm{Bi}(y) \sim \frac{\exp\left[\frac{2}{3}\,(y^{3/2} - x^{3/2})\right]}{2\pi\,(x\,y)^{1/4}} \cdot \left[1 + \frac{5}{48}\left(\frac{1}{y^{3/2}} - \frac{1}{x^{3/2}}\right) + \cdots\right] \tag{42}$$

which are valid for large values of their arguments. Also, the relation

$$\zeta^{(0)} + \int_{\xi_2}^{\xi_3} \Phi^{(0)}(\tau)\,\mathrm{Ai}(\tau)\,\mathrm{d}\tau \cong -\int_{\xi_3}^{\xi_4} \psi^{(0)}(\tau)\,\mathrm{Ai}(\tau)\,\mathrm{d}\tau \ll 1 \tag{43}$$

corresponding to the boundary condition

$$\gamma_{\mathrm{III}}{}'^{(0)}\,(3\,\bar{\eta}_1) = \gamma_{\mathrm{IV}}{}'^{(0)}\,(3\,\bar{\eta}_1)$$

was used in the solution for subregion R_{III} to get a precise result.

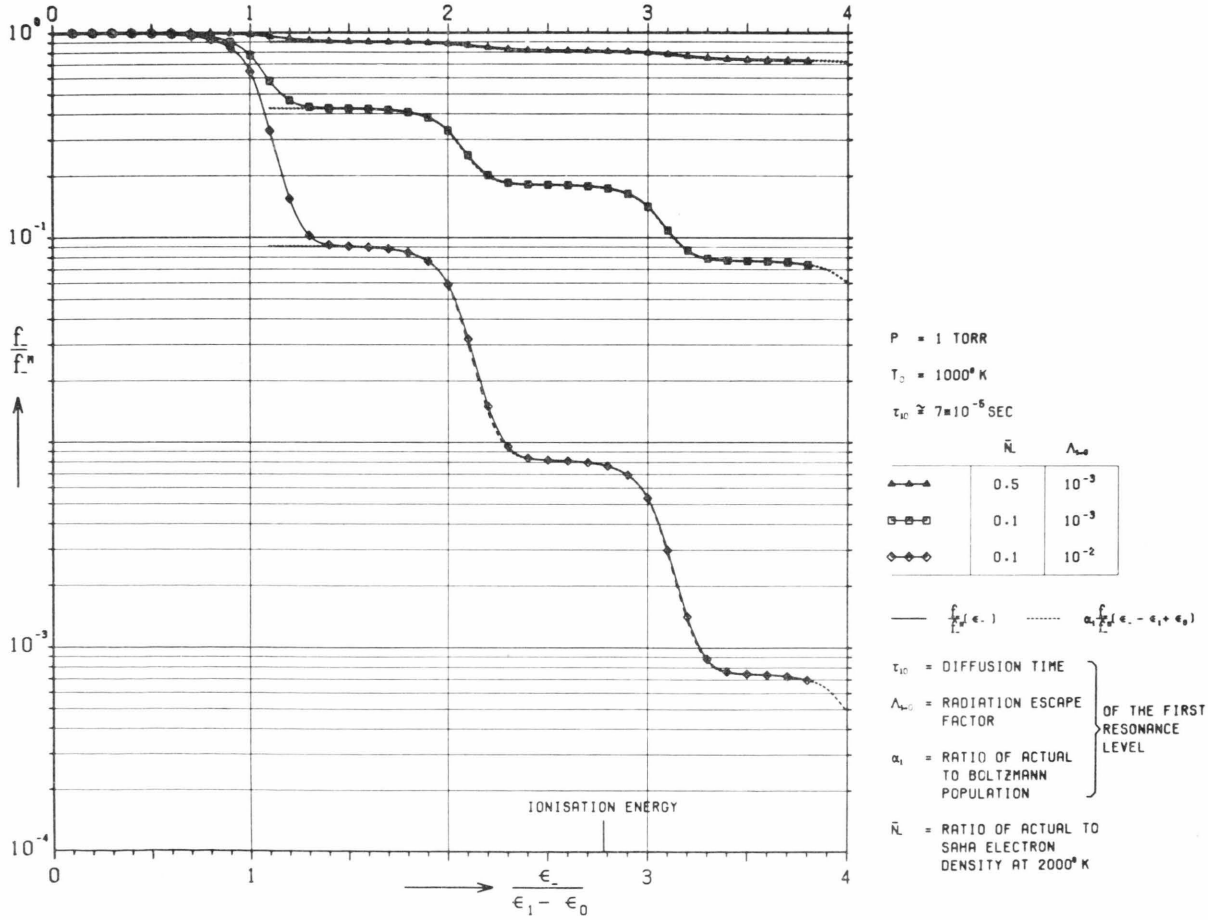

Fig. 3. Disturbance of the electron distribution in a cesium plasma due to resonance radiation escape and deviations from Saha equilibrium.

XI. Results and Discussion

The expressions for the various approximations of the electron distribution function, resonance level population density, and ground state ionization coefficient, all normalized to their equilibrium values, have been numerically evaluated for a cesium plasma. In the calculations electron and gas temperature, total pressure, and diffusion time of the resonance level were kept constant. The results are presented for different parameter combinations of the resonance radiation escape factor $\Lambda_{1\to 0}$ and the ratio $\bar{N}_-$ between actual and Saha electron density at an electron temperature of $T_- = 2000$ K. The parameter values are chosen to correspond to a typical plasma state in an ignited thermionic cesium diode. As shown by the author [28], the assumed values for the resonance radiation escape factor are characteristic for linear plasma dimensions between fractions of a millimeter and several centimeters. The diffusion time of the resonance level was estimated on the assumption that its inhomogeneity length is an order of magnitude larger than its mean free path.

In Fig. 3, the zero order approximation of the normalized electron distribution function, $\gamma^{(0)}(\eta)$, and the corresponding recursion formula, $\alpha_1^{(0)} \cdot \gamma^{(0)}(\eta - \bar{\eta}_1)$, are plotted. The first order approximation of the analytical result, $\gamma^{(1)}(\eta)$, nearly coincides with the zero order approximation in the energy range $\eta \leqq 2\bar{\eta}_1$ and is not shown here. One observes that the zero order approximation agrees very well with the corresponding recursion formula down to $\eta = 1.5\,\bar{\eta}_1$. Slight deviations occur near multiples of the resonance excitation energy, where electron diffusion in energy space is pronounced. Furthermore, one sees that the loss of high energy electrons is sensitive to changes in the electron density and the resonance radiation escape factor. Although it is not shown here, the diffusion time of the resonance level influences the solution in the same manner as the escape factor. This follows directly from the governing equations, which depend on

$$\bar{\alpha}_1 \sim \bar{N}^2_-/(1/\tau_{1d}\,\Lambda_{1\to 0} + \Lambda_{1\to 0}) \text{ and } \varkappa_1 \sim 1/\bar{N}_- \quad (44)$$

for given electron temperature and ground state population density.

In Fig. 4, the normalized expressions of resonance level population, $\alpha_1^{(0)}$, and ground state ionization coefficient, $K^{(0)}_{0\to+}/K^{M}_{0\to+}$, are presented as functions of $\bar{N}_-$. Because of the strong dependence of the electron distribution function on $\bar{N}_-$, these curves give an indication of the interdependence of internal and translational nonequilibrium. There are, however, quantities, which vary only slowly with departures from equilibrium. As an example, the ratio between resonance and ground state ionization rates is plotted as function of $\bar{N}_-$ giving a rough estimate of the cumulative ionization probability.

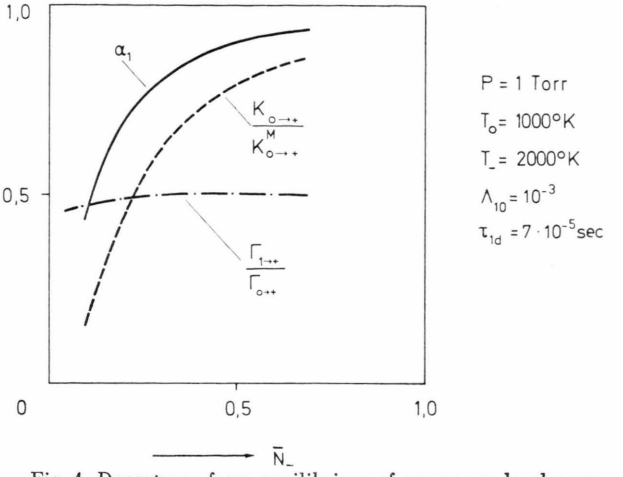

Fig. 4. Departures from equilibrium of resonance level population (solid line), ground state ionization coefficient (dashed line), and ratio between resonance and ground state ionization rate (solid-dotted line) in a cesium plasma.

One of the key assumptions in this work is that of the existence of a low electron temperature. This provides for a moderate ionization degree and determines many of the special features of the solution procedure. Among these are the applicability of the iterative scheme, the excellent agreement between analytical solution and recursion formula down to relatively low electron energies, and the sensitivity of electron distribution and resonance level population to changes in the characteristic parameters.

The present solution is incomplete as far as the electron density is treated as a free parameter and a weak coupling of the resonance state to higher levels is assumed. A selfconsistent calculation including the coupling between all atomic energy levels would, of course, result in an interdependence of the electron density and the various radiation escape factors and diffusion times of the plasma. In particular, $\bar{N}_- = 1$ presupposes $\Lambda_{1-0} = 0$ and $1/\tau_{1d} = 0$ *.

* For a cesium plasma, calculations of $\bar{N}_- < 1$ for $\Lambda_{1\to 0} = 0$ and $1/\tau_{1d} = 0$ have been performed by Norcross and Stone [29]. They assumed a Maxwellian electron distribution for their calculations, which is consistent with our plasma model.

An extension of this work is in progress. It incorporates a selfconsistent determination of the electron density based on a simplified atomic level structure.

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